International Journal of Theoretical Physics, Vol. 22, No. 2, 1983

Affine Mączyński Logics on Compact Convex Sets of States

J. Pykacz

Institute of Mathematics, University of Gdańsk, 80-952 Gdańsk, Poland

Received July 19, 1981

Sets of affine functions satisfying Mączyński orthogonality postulate and defined on compact convex sets of states are examined. Relations between affine Mączyński logics and Boolean algebras when the set of states is a Bauer simplex (classical mechanics, some models of nonlinear quantum mechanics) are studied. It is shown that an affine Mączyński logic defined on a Bauer simplex is a Boolean algebra if it is a sublattice of a lattice consisting of all bounded affine functions defined on the simplex.

1. INTRODUCTION

Mączyński (1973b, 1974) intensively studied sets of functions L, defined on the set of all states S and taking values in the interval $[0,1]$, that satisfy the orthogonality postulate:

> For every orthogonal sequence of functions $\{f_i\} \subseteq L$ there exists a function $f \in L$ such that $f + f_1 + f_2 + \cdots = 1_S$,

where the sequence $\{f_i\} \subseteq L$ is called orthogonal if $f_i(x) + f_j(x) \leqslant 1$ for all $i \neq j$ and all $x \in S$ and a one-element sequence is by definition orthogonal. 0_S and 1_S denote throughout the paper the constant functions equal 0 and 1 for all $x \in S$, respectively. The importance of sets of functions satisfying the orthogonality postulate follows from the theorem proved by Mączyński (1973b): Every set of functions $L \subseteq [0,1]^S$ that satisfies the orthogonality postulate is an orthomodular σ-orthocomplemented partially ordered set admitting full set of probability measures with respect to the natural order ($f \leqslant g$ iff $f(x) \leqslant g(x)$ for all $x \in S$) and complementation $f' = 1 - f$ and conversely, for every orthomodular σ-orthocomplemented partially ordered set L' with a full set of probability measures there exists a set of functions L

0020-7748/83/0200-0097$03.00/0

satisfying the orthogonality postulate and isomorphic to L'. Thanks to these properties any function $f \in L$ can be interpreted as a probability distribution induced on S by a pair (A, E) consisting of an observable A and a Borel subset E of the real line in such a way that $f(x) = p(A, x, E)$ for every $x \in S$ and $p(A, x, E)$ is the probability that a measurement of A will lead to a value in E when the system is in the state x (Mackey, 1963). Following the terminology used in the quantum logic approach to quantum mechanics we adopt the following definition:

Definition 1. A set of functions $L \subseteq [0, 1]^S$ satisfying the orthogonality postulate is called a Mączyński logic.

In the original Mączyński works (1973b, 1974) there are no other restrictions imposed on the set of functions L as well as on the set of physical states S. In our work, however, the sets of states S are compact convex subsets of a Hausdorff locally convex real topological vector space (shortly: compact convex sets) and the sets of functions $L \subseteq [0, 1]^S$ satisfying the orthogonality postulate consist of affine functions only. The assumption about convexity of the set S follows from the fact that we claim any convex combination of states to be again a state of the system. Extreme points of the convex set S obviously represent pure states. The subset of S consisting of all its extreme points will be denoted by *exS*. If a function f is to be interpreted as giving numerical results of measurements of any measurable physical quantity then the function f should be affine since values that f takes on mixed states should be compatible with values that f takes on their components. The base of the locally convex Hausdorff topology τ of the vector space V surrounding S, that we work with, consists of all sets of the form

$$\mathfrak{U}(x_0; A_1, A_2, \ldots, A_n; \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \bigcap_{i=1}^{n} \{x \in V: |\langle A_i, x\rangle - \langle A_i, x_0\rangle| < \varepsilon_i\} \tag{1}$$

where $x_0 \in V$, $\{A_1, \ldots, A_n\}$ is a finite set of measurable physical quantities, $\langle A_i, x\rangle$ for $x \in S$ denotes the mean value of the quantity A_i when the system is in the state x, and $\langle A_i, \cdot\rangle$ is the unique linear extension to the space V of the affine functional defined on S in such a way (see Gudder, 1979, Posiewnik and Pykacz, 1981). If $x, x_0 \in S$ then the numbers ε_i are interpreted as experimental errors made when one measures mean values of A_is. The τ compactness of the set S or *exS* follows from finiteness of sequences of measurements that one can do in reality investigating the set S or *exS*. For details of these considerations we refer the reader to the works of Gunson (1967) or Posiewnik and Pykacz (1981).

Convexity and compactness of the set of all states S together with compactness of the set of all pure states exS allow one to make use of the Choquet theory and represent any state $x \in S$ in the form of a vector integral with respect to a Radon probability measure μ_x concentrated on the set of extreme points of S:

$$x = \int_{exS} y \, d\mu_x \tag{2}$$

The formula (2) generalizes finite or countable convex combinations of pure states to the more realistic infinite case. Generally a measure μ_x that describes the decomposition of a mixed state x into pure ones is not unique; however, there is an important class of compact convex sets—infinite-dimensional simplexes for which the uniqueness occurs. It is commonly accepted (see, for, example Mielnik, 1974) that mixed states in the classical statistical mechanics can be uniquely decomposed into pure ones. Mixed states in some nonlinear models of quantum mechanics have the same feature (Haag and Bannier, 1978) but the nonuniqueness of decomposition of any mixed state into pure ones in the linear quantum mechanics seems again to be one of the crucial attributes of this theory (Mielnik, 1974). From the τ compactness of the set exS it follows that exS is τ closed, i.e., if a set of states is a simplex it is a Bauer simplex. We refer the reader to the book of Alfsen (1971) for all notions concerning the Choquet theory.

2. AFFINE MĄCZYŃSKI LOGICS ON COMPACT CONVEX SETS

It is obvious that the orthogonality postulate cannot be satisfied by an arbitrary set of functions. The most simple examples of (not necessarily affine) Mączyński logics are the two-element logic $L_2(X) = \{0_X, 1_X\}$ and the four-element logic $L_4(X) = \{0_X, f, f' = 1_X - f, 1_X\}$, where X denotes the arbitrary set. The more sophisticated examples of Mączyński logics are not so easy to find, especially when the set X that functions are defined on is multidimensional. The following theorem may be of some help to it.

Theorem 1. Let S be a compact convex set such that exS is closed. The set $L_c \subseteq [0, 1]^S$ consisting of all affine functions that take on the extreme points of S values 0 or 1 only is a Mączyński logic.

Proof. Since the constant function 0_S belongs to L_c and together with any function $f \in L_c$ the function $f' = 1_S - f$ belongs to L_c, it is sufficient to show that the sum of any orthogonal sequence of functions $F = \{f_1, f_2, \ldots\} \subseteq L_c$ belongs to L_c (see Mączyński, 1974). We denote by Y_i a subset of exS

on which a function $f_i \in F$ does not vanish:

$$Y_i = \{y \in exS: f_i(y) \neq 0\} = \{y \in exS: f_i(y) = 1\} \tag{3}$$

The orthogonality of the sequence F implies disjointness of any pair of sets Y_i, Y_j when $i \neq j$. For any $x \in S$ we obtain with the aid of Choquet's theorems

$$\sum_{i=1}^{\infty} f_i(x) = \sum_{i=1}^{\infty} \int_{exS} f_i(y)\, d\mu_x = \sum_{i=1}^{\infty} \int_{exS} d\mu_x = \sum_{i=1}^{\infty} \mu_x(Y_i)$$
$$= \mu_x\left(\bigcup_{i=1}^{\infty} Y_i\right) \leqslant \mu_x(exS) = 1 \tag{4}$$

and since all functions $f_i \in F$ are nonnegative, the sum $\sum_{i=1}^{\infty} f_i(x)$ is an affine function. For any $y \in exS$ we have:

$$\sum_{i=1}^{\infty} f_i(y) = \begin{cases} 0 & \text{if } \forall_{f_i \in F} f_i(y) = 0 \\ 1 & \text{if } \exists_{f_i \in F} f_i(y) = 1 \end{cases} \tag{5}$$

so the sum $\sum_{i=1}^{\infty} f_i$ belongs to L_c and the proof is finished. ∎

Theorem 1 can also be proved when one assumes that S is metrizable instead that exS is closed, but physical justification of such an assumption seems to be much more difficult than physical justification of closedness of exS.

According to the interpretation of classical statistical mechanics in terms of quantum logic notions (Mackey, 1963) every observable A is represented by a real Borel function F_A defined on the phase space Γ, every state x is represented by a probability measure α_x on Γ, and for any Borel subset E of the real line

$$p(A, x, E) = \alpha_x[F_A^{-1}(E)] \tag{6}$$

Pure states are represented by probability measures concentrated on one-point subsets of Γ (Dirac measures), so if we follow Mączyński (1974) and assume that every function $f \in [0, 1]^S$ is generated by a pair (A, E) in such a way that

$$f(x) = f_{(A, E)}(x) = p(A, x, E) \tag{7}$$

then for any pure state $y \in exS$ represented by a Dirac measure $\delta_{\tilde{y}}$ concentrated on the one-point subset $\{\tilde{y}\} \subset \Gamma$ we obtain

$$f(y) = f_{(A,E)}(y) = \delta_{\tilde{y}}[F_A^{-1}(E)] = \begin{cases} 0 & \text{if } \tilde{y} \notin F_A^{-1}(E) \\ 1 & \text{if } \tilde{y} \in f_A^{-1}(E) \end{cases} \tag{8}$$

therefore Theorem 1 can be applied in the case of classical statistical mechanics to affine functions that map a set of states onto [0, 1] as soon as the assumption (7) is fulfilled. We shall examine affine Mączyński logics defined on sets of classical states more carefully in the following section.

3. AFFINE MĄCZYŃSKI LOGICS ON BAUER SIMPLEXES

There are two main properties that distinguish classical statistical systems from quantum systems:

(i) In a classical statistical mechanics all observables are simultaneously measurable.

(ii) Every state of a classical statistical system can be uniquely decomposed into pure states.

Property (i) can be mathematically expressed by saying that the logic of a classical system is a Boolean algebra (Mackey, 1963), whereas property (ii) suggests that the set of states of a classical statistical system is an infinite-dimensional simplex (Mielnik, 1974; Posiewnik and Pykacz, 1981) and from the topological considerations it follows that it is exactly a Bauer simplex, i.e., the set of its extreme points is closed. Consequently, an affine Mączyński logic defined on a Bauer simplex should be a Boolean algebra. Since we deal with functions that map a set of states into [0, 1] we adopt the following definition:

Definition 2. Let $A \subset [0, 1]^S$ be a set of functions from $S \neq \emptyset$ into [0, 1]. We say that A is a numerical Boolean algebra if A is a Boolean algebra with respect to the natural order ($f \leq g$ iff $f(x) \leq g(x)$ for all $x \in S$) and complementation $f' = 1_S - f$ and if $f \vee g = f + g$ for $f \wedge g = \wedge_S = 0_S$.

The notion of a numerical Boolean algebra was introduced by Mączyński (1973a). He showed that every Boolean algebra can be isomorphically represented as a numerical Boolean algebra.

Every σ-complete Boolean algebra is an orthomodular σ-orthocomplemented poset admitting a full set of probability measures; therefore from the Mączyński (1973b) theorem one obtain the following corollary:

Corollary 1. Every σ-complete Boolean algebra can be isomorphically represented as a Mączyński logic. Every σ-complete numerical Boolean algebra is a Mączyński logic.

The second part of Corollary 1 follows from constructions presented in Mączyński papers (1973a, 1973b). In our considerations we will use the following theorem, which is also due to Mączyński.

Theorem 2. (Mączyński, 1974; proof in Mączyński, 1973a). A Mączyński logic L is a σ-complete numerical Boolean algebra if and only if for any $f_1, f_2 \in L$ there are g_1, g_2, g_3 in L such that $g_1 + g_2 + g_3 \leqslant 1_S$ and $f_1 = g_1 + g_2, f_2 = g_2 + g_3$.

Let K be a Bauer simplex. The vector space $\mathcal{C}_b(K)$ consisting of all bounded affine functions defined on K is a vector lattice (see, for example, Alfsen, 1971); therefore it is natural to investigate such affine Mączyński logics defined on K which are sublattices of $\mathcal{C}_b(K)$. Indeed, for such logics the following theorem is valid.

Theorem 3. Let K be a Bauer simplex. If a Mączyński logic $L \subseteq [0,1]^K$ is a sublattice of $\mathcal{C}_b(K)$ then it is a σ-complete numerical Boolean algebra.

In the proof of Theorem 3 we shall use the following lemmas.

Lemma 1. If K is a Bauer simplex, then

$$\forall_{f, g \in \mathcal{C}_b(K)} f \leqslant g \Leftrightarrow f|_{exK} \leqslant g|_{exK} \tag{9}$$

Where $F|_{exK}$ denotes the restriction of a function f to the set exK consisting of all extreme points of a simplex K.

Proof. If $f|_{exK} \leqslant g|_{exK}$ then for any $x \in K$ we obtain by the Choquet theorems

$$f(x) = \int_{exK} f\Big|_{exK} d\mu_x \leqslant \int_{exK} g\Big|_{exK} d\mu_x = g(x) \tag{10}$$

The converse implication is obvious. ■

Lemma 2. If K is a Bauer simplex, then for any $f, g \in \mathcal{C}_b(K)$ the following statements are valid:

(i) The meet of f and g in the lattice $\mathcal{C}_b(K)$ is the affine and continuous extension to the whole set K of the function $\min(f, g)$ restricted to the exK.

(ii) The join of f and g in the lattice $\mathcal{A}_b(K)$ is the affine and continuous extension to the whole set K of the function $\max(f, g)$ restricted to the exK.

Proof. The bounded functions f and g are continuous; therefore the function $\min(f, g)$ (which is not in general an affine function) and its restriction to the set exK are also continuous. By the Bauer theorem (see, for example, Alfsen, 1971) every affine and continuous function defined on the extreme points of a Bauer simplex can be uniquely extended to an affine and continuous function defined on a whole Bauer simplex. Let us denote by h such extension of the function $\min(f, g)|_{exK}$. Consequently $h \in \mathcal{A}_b(K)$ and by formula (9) $h \leq f$ and $h \leq g$. If d is an arbitrary bounded affine function such that $d \leq f$ and $d \leq g$, then

$$d|_{exK} \leq f|_{exK}, \qquad d|_{exK} \leq g|_{exK} \tag{11}$$

so $d|_{exK} \leq \min(f, g)|_{exK}$ and by Lemma 1 $d \leq h$. This finishes the proof of part (i). ■

The proof of part (ii) is analgous.

Proof of Theorem 3. We shall show that for any $f_1, f_2 \in L$ there exist functions g_1, g_2, g_3 described in Theorem 2. Since L is a sublattice of the lattice $\mathcal{A}_b(K)$, we can take the meet of f_1 and f_2 as a function g_2:

$$g_2 = f_1 \wedge f_2 \tag{12}$$

Let us notice that for any $x \in exK$ by Lemma 2 we have

$$\begin{aligned} f_1'(x) + (f_1 \wedge f_2)(x) &= 1 - f_1(x) + \min(f_1(x), f_2(x)) \\ &= \begin{cases} 1 - f_1(x) + f_1(x) & \text{if } f_1(x) \leq f_2(x) \\ 1 - f_1(x) + f_2(x) & \text{if } f_1(x) > f_2(x) \end{cases} \end{aligned} \tag{13}$$

Therefore by Lemma 1 $f_1' + f_1 \wedge f_2 \leq 1_K$ and by the orthogonality postulate there exists in the logic L a function g_1 such that

$$g_1 = 1_K - f_1' - f_1 \wedge f_2 = f_1 - g_2 \tag{14}$$

In the similar way we can show that there exists in the logic L a function g_3 such that

$$g_3 = 1_K - f_2' - f_1 \wedge f_2 = f_2 - g_2 \tag{15}$$

Let us notice that

$$\begin{aligned} g_1 + g_2 &= f_1 \leqslant 1_K \\ g_3 + g_2 &= f_2 \leqslant 1_K \end{aligned} \tag{16}$$

and that for any $x \in exK$ we obtain

$$\begin{aligned}(g_1+g_3)(x) &= f_1(x)+f_2(x)-2\min(f_1(x), f_2(x)) \\ &= \begin{cases} f_1(x)+f_2(x)-2f_1(x) & \text{if } f_1(x) \leqslant f_2(x) \\ f_1(x)+f_2(x)-2f_2(x) & \text{if } f_1(x) > f_2(x) \end{cases} \\ &= \begin{cases} f_2(x)-f_1(x) & \text{if } f_1(x) \leqslant f_2(x) \\ f_1(x)-f_2(x) & \text{if } f_1(x) > f_2(x) \end{cases}\end{aligned} \tag{17}$$

so by Lemma 1 $g_1 + g_3 \leqslant 1_K$ and by the orthogonality postulate

$$g_1 + g_2 + g_3 \leqslant 1_K \tag{18}$$

This finishes the proof of Theorem 3. ■

The following example shows that Theorem 3 cannot be reversed since there exist a Bauer simplex K and a Mączyński logic $L \subseteq [0,1]^K$ such that L is a σ-complete numerical Boolean algebra but L is not a sublattice of a lattice $\mathfrak{C}_b(K)$.

Example 1. Let K be a closed interval $[a, b]$, $a \neq b$. The set K, regarded as a compact convex subset of a real line equipped with the natural topology, is obviously a Bauer simplex. The set consisting of affine functions 0_K, 1_K, f and $f' = 1_K - f$, where

$$\begin{gathered} f(a)=0, \qquad f(b)=\alpha \in (\tfrac{1}{2}, 1) \\ f[pa+(1-p)b] = pf(a)+(1-p)f(b) = (1-p)\alpha \quad \forall_{p \in [0,1]} \end{gathered} \tag{19}$$

is the four-element affine Mączyński logic $L_4(K)$. It can easily be checked that $L_4(K)$ is a lattice in which lattice operations $\wedge_L$ and $\vee_L$ are defined as follows:

$$\begin{aligned} & 0_K \wedge_L f = 0_K \wedge_L f' = 0_K \wedge_L 1_K = f \wedge_L f' = 0_K \\ & f \wedge_L 1_K = f, \qquad f' \wedge_L 1_K = f' \\ & 1_K \vee_L f = 1_K \vee_L 0_K = f \vee_L f' = 1_K \vee_L f' = 1_K \\ & f \vee_L 0_K = f, \qquad f' \vee_L 0_K = f' \end{aligned} \tag{20}$$

By the immediate checking Definition 2 or assumptions of Theorem 2 it can be shown that $L_4(K)$ is a σ-complete numerical Boolean algebra but the join $f \vee f'$ of f and f' in the lattice $\mathcal{Q}_b(K)$ is an affine extension onto K of the function that takes value 1 on the point a and value α on the point b; therefore

$$(f \vee f')(b) = \alpha \neq 1 = (f \vee_L f')(b) \tag{21}$$

so the logic $L_4(K)$ is not a sublattice of a lattice $\mathcal{Q}_b(K)$.

The logic in Example 1 is a lattice and it is a numerical Boolean algebra. The next example shows, however, that not every affine Mączyński logic defined on a Bauer simplex that is a lattice is a numerical Boolean algebra.

Example 2. Let K be the same set as in Example 1 and let add to the logic that was described in the previous example two functions g and g' such that $g(x) = g'(x) = 1/2$ for any $x \in K$. It is easy to check that the set $L = L_4(K) \cup \{g, g'\}$ obtained in such a way is again an affine Mączyński logic in which $\{f, f'\}$ and $\{g, g'\}$ are the only nontrivial orthogonal sequences. L is a lattice but it is not a sublattice of $\mathcal{Q}_b(K)$ for the same reasons as a logic in the Example 1. L is not a numerical Boolean algebra since $f \wedge_L g = 0_K$ but $f \vee_L g = 1_K \neq f + g$.

Finally we shall examine the affine Mączyński logic described in Theorem 1 in the case when it is defined on a Bauer simplex. We shall show that it fulfills assumptions of Theorem 3 so we obtain the following corollary.

> *Corollary 2.* Let S be a Bauer simplex. The set $L_c \subseteq [0,1]^S$ consisting of all affine functions that take on the extreme points of S values 0 or 1 only is a σ-complete numerical Boolean algebra.

Proof. Let $\vee_{L_c}$, $\wedge_{L_c}$ and $\vee$, $\wedge$ denote lattice operations in the logic L_c and in the lattice $\mathcal{Q}_b(S)$, respectively. For any $f, g \in L_c$ the affine functions $f \wedge g$ and $f \vee g$ belong to L_c since by Lemma 2 for any $x \in exS$ we have

$$\begin{aligned}(f \wedge g)(x) &= \min(f(x), g(x)) \in \{0, 1\} \\ (f \vee g)(x) &= \max(f(x), g(x)) \in \{0, 1\}\end{aligned} \tag{22}$$

Of course $f \wedge g \leqslant f, g \leqslant f \vee g$, which implies inequalities

$$f \wedge g \leqslant f \wedge_{L_c} g, \qquad f \vee_{L_c} g \leqslant f \vee g \tag{23}$$

but $L_c \subset \mathcal{E}_b(S)$; therefore we have also

$$f \wedge_{L_c} g \leqslant f \wedge g, \qquad f \vee g \leqslant f \vee_{L_c} g \tag{24}$$

From the formulas (23) and (24) we obtain $f \wedge_{L_c} g = f \wedge g$, $f \vee_{Lc} g = f \vee g$. This finishes the proof.

Corollary 2 shows that the set L_c satisfies main conditions for being a model of a logic associated with a classical system. The index "c"="classical" in L_c was chosen because of this fact.

REFERENCES

Alfsen, E. M. (1971). *Compact Convex Sets and Boundary Integrals*. Springer, Berlin.

Gudder, S. P. (1979). Axiomatic operational quantum mechanics, *Reports on Mathematical Physics*, **16**, 147–166.

Gunson, J. (1967). On the algebraic structure of quantum mechanics, *Communications in Mathematical Physics*, **6**, 262–285.

Haag, R., and Bannier, U. (1978). Comment on Mielnik's generalized (nonlinear) quantum mechanics, *Communications in Mathematical Physics*, **60**, 1–6.

Mackey, G. (1963). *Mathematical Foundations of Quantum Mechanics*. Benjamin, New York.

Mączyński, M. J. (1973a). On some numerical characterisation of Boolean algebras, *Colloquium Mathematicum*, **27**(2), 207–210.

Mączyński, M. J. (1973b). The orthogonality postulate in axiomatic quantum mechanics, *International Journal of Theoretical Physics*, **8**(5), 359–360.

Mączyński, M. J. (1974). Functional properties of quantum logics, *International Journal of Theoretical Physics*, **11**(3), 149–156.

Mielnik, B. (1974). Generalized quantum mechanics, *Communications in Mathematical Physics*, **37**, 221–256.

Posiewnik, A., and Pykacz, J. (1981). Choquet properties of the set of physical states, University of Gdańsk, Institute of Physics, preprint No. 13.